Some Classes of Diophantine Edge Graceful graph

ABSTRACT

In this paper we consider only finite graphs. A graph G (V, E) is said to be *Diophantine edge graceful graph*, if there exists a bijection $: E(G) \rightarrow \{1,2,3, \ldots \ldots .. q\}$ such that the labels of every set of m edges satisfy a linear Diophantine equation. In [2], we constructed Diophantine equations and used the solutions for edge graceful trees of the form (m, h) trees. In this paper we extend the results of [2] to Ramanujan graphs. A d-regular graph is called Ramanujan graph, if the absolute value of second highest adjacency eigen value is bounded above by 2. A graph is graceful if its m edges are labelled by a unique positive integer from 1 to m consecutively. A major unproven-conjecture in graph theory is the graceful tree conjecture or Ringel-Kotzig conjecture, which hypothesizes that all trees are graceful. For the class of complete (m, h) trees, the authors [2] proved that they are Diophantine edge graceful.

The main results of this paper are *Diophantine edge graceful graph* of complete graphs k_n (n odd) and the n-prism graphs. It is interesting to observe that these two classes of graphs are also Ramanujan graphs and the solutions of the corresponding Diophantine equations again yield edge-graceful graphs.

Keywords:
Diophantine edge graceful graph, Ramanujan graph .

1. INTRODUCTION

Consider a graph $G = (V, E)$, with $|V| = p$ and $|E| = q$. If $f: V \rightarrow \{0,1,2, \ldots .. p\}$ is a bijective mapping and if $f^+: E \rightarrow \{1,2, \ldots . q\}$ be defined by $f^+(uv) = |f(u) - f(v)|$, for $u, v \in V$, and if f^+ is bijective, then the induced map f^+ gives an edge graceful labeling. A graph G (V, E) is said to be *Diophantine edge graceful graph*, if there exists a bijection $: E(G) \rightarrow \{1,2,3, \ldots \ldots .. q\}$ such that the labels of every set of m edges satisfy a linear Diophantine equation. In this paper, we for any given graph G (may be complete graph, n –prism graph or, a Ramanujan graph), we construct a specific Diophantine equation and solve it and label the vertices consecutively as a graceful Graph. If there are m edges in G, then edges are labelled from 1, 2, 3, ..., m . The labelling gives an edge graceful Graph.

Definition : Let G be a simple graph. Let A(G) = adjacency matrix of G = { (i, j) = 1, if ij is an edge and (i,j) = 0, otherwise.}. If A (G) is real symmetric, then sum of eigen values becomes zero, if not average of eigen values be non-zero, say m.

Definition : Let G be a simple graph, then Energy of G = E (G) = sum of absolute differences between an eigen value and the average m of the eigen values.

But if the average is zero, in the case of A(G) is a real symmetric Matrix, then E(G)=sum of absolute values of eigen values of A(G), because in this case, m=0..

We observe that every complete graph Kn is (n-1)-regular graph. If G is a d-regular graph, then it is called a Ramanujan graph. If L is the modulus of the highest eigen value of A (G) = adjacency matrix of G and if L ≤ $2\sqrt{d-1}$. Also the n-prism graph G is a 3-regular graph. The class of generalised Petersen graphs, n-Petersen graph, for n= odd positive integer, turns out to be Ramanujan Graphs and also have minimal energy.

In section 1, we get results concerning those graphs that are Ramanujan graphs which admit Diophantine edge graceful labelling this includes the explicit construction of the Diophantine equations, relevant to the chosen graph G. The interesting feature is that the solutions of Diophantine equations are labelled for the different Hamiltonian cycles in G. Theorem 1, concerns the complete graphs $G = K_n$ for odd Theorem 2, concerns the technique of Diophantine edge labelling of all n-prisms.

In section 2, we study Diophantine edge graceful labelling of Generalised n-Petersen graphs. If G is an n-prism and if P is a Generalised Petersen graph P, the special property: energy of (P) ≤ Energy (G), where G is any n-prism is given in theorem 3.

In section3: theorem 4 proves edge-gracefulness for the complete bipartite graph $K_{n,m}$, that are not necessarily Ramanujan graphs and we have also results on Wheel graphs W_n proved.

2. SECTION ONE

In this section , we consider the Diophantine edge graceful graph which is also Ramunajan graphs.

Definition: A complete digraph is a directed graph in which every pair of distinct vertices is connected by a pair of unique edges .

THEOREM 1:
Every complete graph K_n (n odd) is a Diophantine edge graceful. The Diophantine equation satisfied by Hamiltonian cycles of K_n is

$$2x_1 + \sum_{i=2}^{n} x_i = ns^2 + (s+1)^2 \text{ -------------------- (1)}$$

Where s is the number of Hamiltonian cycles in K_n (n odd).

Proof: We know that for a complete graph K_n (n odd), we have (n-1)/2 Hamiltonian cycles.
Consider a complete graph K_n, with n=2k+1, k >0 vertices. Then it will have k number of Hamiltonian cycles.
We have to prove that every k Hamiltonian cycles with n vertices satisfy the Diophantine equation (1).
We observe that in every K_n, with k Hamiltonian cycles have nk edges. We divide nk edges into k number of equation s having n variables. Also we multiply 2 to every first variables so that each k equation with n edges satisfy (1)
.
(i.e.) $2x_1 + \sum_{i=2}^{n} x_i$ will satisfy the Diophantine equation as follows

$$= \left[[2(1+2+\cdots+k)] + \sum_{i=k+1}^{nk} x_i\right]/k$$

$$= \left[\frac{2k(k+1)}{2} + \frac{nk(nk+1)}{2} - \frac{k(k+1)}{2}\right]/k$$

$$= \frac{1}{2}[k+1+(2k+1)\{(2k+1)k+1\}]$$

$$= \frac{1}{2}[k+1+(2k+1)^2 k + (2k+1)]$$

$$= \frac{1}{2}[k+1+4k^3+4k^2+k+2k+1]$$

$$= \frac{1}{2}[4k^3+4k^2+4k+2]$$

$$= 2k^3+2k^2+2k+1$$

$$= 2k^3+k^2+k^2+2k+1$$

$$= (2k+1)k^2+(k+1)^2$$

$$= ns^2+(s+1)^2.$$

Hence the theorem.

Example 1:

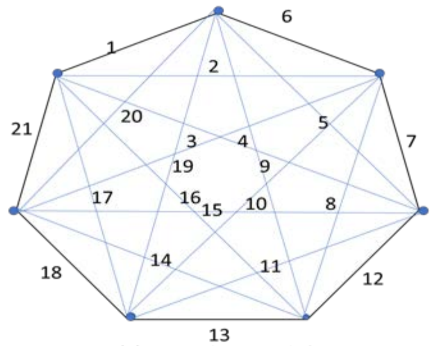


Fig1 (a) : Complete graph for n=7

In K_7, we have 3 Hamiltonian cycles. Each Hamiltonian cycle satisfy the Diophantine equation:

$$2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 79$$

Here s=3, n=7 then $7 \times 3^2 + (3+1)^2 = 79$

OBSERVATIONS:

1. Every complete graph K_n (n odd), the Diophantine equation can be given in the form

$$2x_1 + \sum_{i=2}^{n} x_i = \frac{1}{4}(n^3 - n^2 + 3n + 1)$$

2. For every complete graph K_n (n odd), the Diophantine equation also satisfy $2x_1 + \sum_{i=2}^{n} x_i = \pm 1 (mod 4)$.

Definition:A prism graph is a graph that has one of the prisms as its skeleton.

THEOREM 2:

Every n-prism graph is Diophantine edge graceful; the n graceful edge labels satisfy the Diophantine equation
(i)When n is even: $\sum_{i=1}^{n} x_i = \frac{n}{2}(3n+1)$
(ii)When n is odd:

$$2x_1 + \sum_{i=2}^{n} x_i = \frac{1}{2}(3n^2 + n + 4) = \frac{n}{2}(3n+1) + 2$$

Proof is similar to theorem 1
Example:
Consider the graph n-Prism for n=4 and n=5

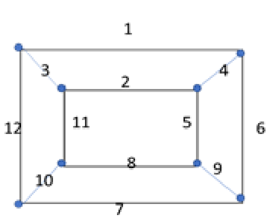


Fig 2(a): n –Prism for n=4

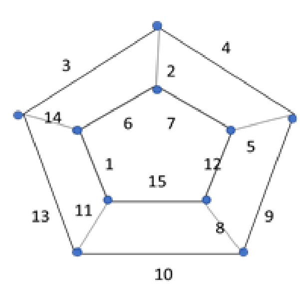


Fig 2(b): n –Prism for n=5

In fig 2(a), the Diophantine equation satisfied by the set of 4 edges is $x_1 + x_2 + x_3 + x_4 = 26$. Also, in fig 2(b), the Diophantine equations satisfied by the set of 5 edges are $2x_1 + x_2 + x_3 + x_4 + x_5 = 42$.

GENERALISED PETERSEN GRAPHS:

The Petersen graph is an undirected graph with 10 vertices and 15 edges. It is a is the complement of the line graph of K_5 . The generalized Petersen graphs are a family of cubic graphs formed by connecting the vertices of a regular polygon to the corresponding vertices of a star polygon. They include the Petersen graph and generalize one of the ways of constructing the Petersen graph. The generalized Petersen graph family was introduced in 1950 by H. S. M. Coxeter and was given its name in 1969 by Mark Watkins.

We observe that Generalised Petersen graphs satisfy the same Diophantine Equations of n- Prisms for each n. For n=5 which is the Petersen Graph satisfy the Diophantine equation same as in example 2(b): $2x_1 + x_2 + x_3 + x_4 + x_5 = 42$.

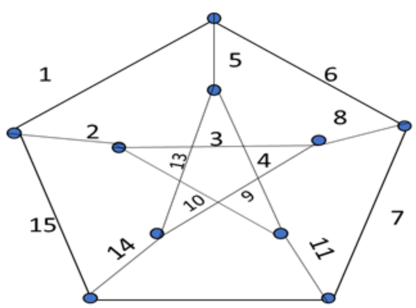


Fig 2(c): Petersen Graph

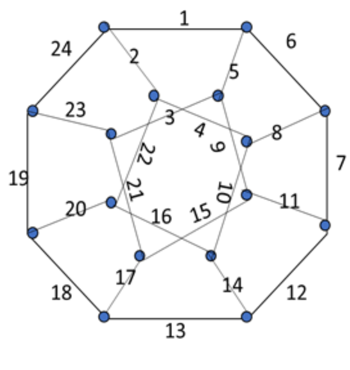


Fig 2(d): Generalised Petersen Graph G(8,2)

If X is a connected k-regular graph, we may arrange the Eigen values as

$$k = \lambda_0(X) \geq \lambda_1(X) \geq \cdots \ldots . \geq \lambda_n(X) = -k$$

It is not difficult to show that $-k$ is an eigen value of X if and only if X is bipartite in which case its multiplicity is again equal to the number of connected components. Any eigen value $\lambda_i \neq \pm k$ is referred to as a nontrivial eigen value. The maximum of the absolute values of all the non-trivial eigen values is denoted as λ(X). A Ramanujan multigraph is a k-regular graph satisfying

$$\lambda(X) \leq 2\sqrt{k-1}. \text{ ------ (1)}$$

Considering the above graphs, i.e. Complete graphs, n-prism graphs and generalized Petersen graphs, we observe that all these graphs are regular and satisfy the condition (1). So all these graphs are also Ramanujan Graphs. Considering the above graphs, we observe that all the graphs satisfy the conditions of Ramanujan Graphs. If G is any simple graph, then Energy(G)= sum of absolute values of eigen values of A(G) = the adjacency matrix of G = {(i ,j) = 1, if (i, j) is an edge in G, and (i, j) = 0, otherwise}.review the main points of the paper, do not replicate the abstract as the conclusion. A conclusion might elaborate on the importance of the work or suggest applications and extensions.

THEOREM 3:

For n a positive odd integer, the n-prism and the Generalised n-Petersen graphs are both 3-regular graphs and they also satisfy the conditions of Ramanujan graphs. If G is an-n prism and P is n-Petersen graph, then Energy (G)>= Energy (P).The n-Petersen graph has minimal energy for n=3, 5, 7, 9...

Example: Let G= a 7-prism and P = a 7-Petersen graph. Let A(G) , A(P) are the adjacency matrices of G and P respectively :then A(G)= [...] and A(P)=[...] be 14x14 adjacency matrices.

Then the eig(A(G)) = spec(A(G))=(-2.8017 -1.445 -0.8019 0.247 0.555 1 2.247 3) each with multiplicity 2 , ,except 1 and 3 occurring once. Evidently, the 7-prism is 3-regular and satisfies the conditions of a Ramanujan Graph. Since average of absolute values of eigen values = 1.4426 = non-zero and Energy(A(G)) = sum(abs((eig(A(G))-1.4426)) = 26.5257.
On the other hand, Eig (P-7petersen) = {3,-2.3319, -2.3028, -2.1007, -2.0000, -0.9089, -0.6180, 0.0000, 0.849, 1,0000, 1.3028, 1.7108, 1.5180, 1.5457} and average of

absolute values of eigen values = 1.455 and Energy =Sum(abs(A(P)-1.466))=24.5451..and E(A(P)) = 4.5451< E(G) = 26.5257.

Thus Energy (Petersen-7 prism) < Energy (7-Prism).and so the Petersen-7 prism has LEAST Energy among 7-Prisms.
In general, Energy of an n-Petersen graph is minimum among that of a general n-prism, for n=3, 5, 7, 9, ...

3. SECTION TWO:

In this section, we consider the graphs which are only Diophantine edge graceful graphs.

Definition: A complete bipartite graph is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. That is, it is a bipartite graph (V_1, V_2, E) such that for every two vertices $v_1 \in V_1$ and $v_2 \in V_2$, v_1v_2 is an edge in E. A complete bipartite graph with partitions of size $|V_1|=m$ and $|V_2|=n$, is denoted $K_{m,n}$

THEOREM 4:

Every complete bipartite graph $k_{m,n}$ is an edge graceful graph: The Diophantine equation satisfied by the mn edges fall in one of the three cases

(i) If m is even, then n equations with m edges satisfy
$\sum_{i=1}^{m} x_i = \frac{m}{2}(mn+1)$
(ii) If n is odd then m equations with n edges satisfy
$\sum_{i=1}^{n} x_i = \frac{n}{2}(mn+1)$
(iii) If both m and n are odd, then m equations with n edges satisfy $\sum_{i=1}^{m} x_i = \frac{1}{2}(mn^2+m+n+1)$, m > 1.

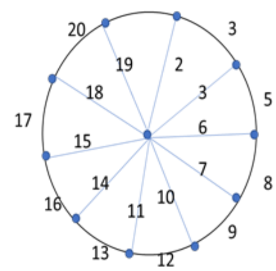
Remark: The Complete Bipartite graph $k_{m,n}$ is not Ramanujan Graph in general, but if m=n ,the graph $k_{n,n}$is a Ramanujan Graph in trivial sense.

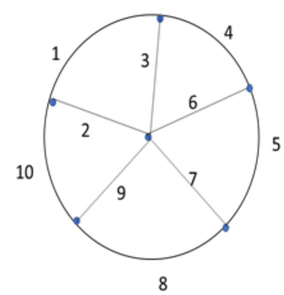
Definition:
A wheel graph is a graph formed by connecting a single universal vertex to all vertices of a cycle.

Theorem 4:

All Wheel graphs W_n are Diophantine edge graceful graphs. The Diophantine equation satisfied by the n edges is:

(a) For n even, we have
$\sum_{i=1}^{m} x_i = \frac{n}{2}(2n+1)$
(b) For n odd, we have
$2x_1 + \sum_{i=2}^{n} x_i = \frac{n}{2}(2n+1) + 3/2$





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CONCLUSION:

Labelled graph has many applications due to which it one of the topics of current interest. Here we have given five classes of graphs which are Diophantine edge graceful. This can be carried out for other classes of Graph also.

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